Algebraic elements for the notion of ‘many’

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Abstract

Considering a first-order logic extended by a new quantifier, proposed to interpret the notion of ‘many’, we investigate the possibility to keep the same mathematical structure that interprets the natural quantifiers into a modal propositional logic. Then, we introduce the concept of upper closed set and using algebraic tools we obtain a new modal logical system with the correspondent adequate algebraic semantic.

Introduction

Natural languages present distinct quantifiers from the logical quantifiers ‘universal - $\forall$’ and ‘existential - $\exists$’ which cannot be defined from these, but that deserve to be investigated in formalized and mathematical contexts. Examples of these quantifiers are the expressions: many, almost all, too many individuals, among others.

Sette, Carnielli and Veloso (1999) and Carnielli and Veloso (1997) introduced the Ultrafilter Logic with the aim of formalizing propositions of the kind ‘almost all’ or ‘generally’. This logic has as model a classical first-order structure $\mathcal{A}$, endowed with an ultrafilter defined over the universe of $\mathcal{A}$.

Motivated by this work, Grácio (1999) introduced the family of modulated logics and, among them, the Logic of Many that has the meaning of 'many' interpreted by the concept of proper family of upper closed sets, as we will see below.

In this work, we investigate the underlying algebraic structures which allow an algebraic version of the concept of 'many'. We call these algebraic structures by 'upper closed sets' and we use them in a similar way to the algebraic version of filters and ultrafilters in a lattice.

First, we present the Logic of Many, a particular case of modulated logics, as introduced in Grácio (1999). Next we develop algebraic elements of upper closed sets that will make possible the algebraic developments of the algebra of 'many'. Then, we introduce a modal propositional logic in which the modal operator will be interpreted into an algebraic structure that involves the concept of upper closed set in a Boolean algebra. By means of algebraic resources, we show the soundness and the completeness of this modal propositional logic.

1 The Logic of Many

In this section we look at the particularization of modulated logics, the Logic of Many, that motivated the development of this paper.

1.1 Proper family of upper closed sets

Grácio (1999) considers that the following properties must be contemplated when we propose to formalize some notion of 'many':

(i) If φ is a true sentence for all individuals of the universe, then φ is true for many individuals of this universe;

(ii) If φ is a true sentence for many individuals, then there is someone in the universe that satisfies φ;

(iii) If the set of individuals that satisfy φ is included in the set of individuals that satisfy ψ and there are many individuals that satisfy φ, then there are many individuals that satisfy ψ too.

This way, the 'vague' notion of 'many' approached here is associated to the concept of large set of evidences, but not necessarily linked to the cardinal notion of majority.

The notion of 'many' exposed above can be captured by the mathematical concept of proper family of upper closed sets.

A *proper family of upper closed sets* over an universe E is a collection $\mathcal{F}$ of subsets of E such that:

(i) $B \in \mathcal{F}$ and $B \subseteq C \Rightarrow C \in \mathcal{F}$;

(ii) $E \in \mathcal{F}$;

(iii) $\emptyset \notin \mathcal{F}$.

1.2 A logical system for 'many'

Let $\mathcal{L}$ be the classical first-order logic with equality. The Logic of Many, denoted by $\mathcal{L}(M)$, is obtained from $\mathcal{L}$ in the following way.

The axioms of $\mathcal{L}(M)$ are those of $\mathcal{L}$ plus the following additional axioms for the new quantifier (of many) M:

(Ax_1) $\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow (Mx\varphi(x) \leftrightarrow Mx\psi(x))$

(Ax_2) $Mx\varphi(x) \rightarrow My\varphi(y)$, when y is free for x in $\varphi(x)$

(Ax_3) $\forall x\varphi(x) \rightarrow Mx\varphi(x)$

(Ax_4) $Mx\varphi(x) \rightarrow \exists x\varphi(x)$

(Ax_5) $\forall x(\varphi(x) \rightarrow \psi(x)) \rightarrow (Mx\varphi(x) \rightarrow Mx\psi(x))$.

The deduction rules of the system $\mathcal{L}(M)$ are the same defined for $\mathcal{L}$, specifically:

(MP) *Modus Ponens*: φ, $\varphi \rightarrow \psi$ / ψ

(Gen) Generalization: φ / $\forall x\varphi(x)$.

The axioms $A_3 - A_5$ are specific of the Logic of Many. The two first axioms have the task of making possible the logical adequacy with the proposed model for this logic.

The original intuition for these axioms is given by clauses (i) - (iii) above.

The usual syntactical notions for $\mathcal{L}(M)$, such as sentence, proof, theorem, logical consequence, consistence and others are defined in an analogue way as in the classical first-order logic $\mathcal{L}$.

1.3 Semantics for the Logic of Many

Let $\mathcal{A}$ be a classical first-order structure with universe A. A *structure of proper family of upper closed sets*, denoted by $\mathcal{A}^{\mathcal{F}}$, for $\mathcal{L}(M)$ is obtained from the structure $\mathcal{A}$ endowing it with a proper family of upper closed sets $\mathcal{F}$ over the universe A.

The interpretation of relation, function and constant symbols is the same as in $\mathcal{L}$ with respect to $\mathcal{A}$.

The notion of *satisfaction* of a formula of $\mathcal{L}(M)$, in a structure $\mathcal{A}^{\mathcal{F}}$, is inductively defined in usual way by adding the following clause:
• let φ be a formula whose free variables set is contained in $\{x\} \cup \{y_1, ..., y_n\}$ and $\overline{a} = (a_1, ..., a_n)$ a sequence of elements of A. Then:

$$\mathcal{A}^{\mathcal{F}} \vDash Mx\varphi[x, \overline{a}] \Leftrightarrow \{b \in A : \mathcal{A}^{\mathcal{F}} \vDash [b, \overline{a}]\} \in \mathcal{F},$$

in which, as usual, $\mathcal{A}^{\mathcal{F}} \vDash Mx\psi[\overline{a}]$ denote $\mathcal{A}^{\mathcal{F}} \vDash_s \psi$, when the free variables of the formula ψ belong to the set $\{z_1, ..., z_n\}$, $s(z_i) = b_i$ and $\overline{a} = (a_1, ..., a_n)$.

For a sentence $Mx\varphi(x)$, we have:

$$\mathcal{A}^{\mathcal{F}} \vDash Mx\varphi(x) \Leftrightarrow \{a \in A : \mathcal{A}^{\mathcal{F}} \vDash \varphi(a)\} \in \mathcal{F}.$$

Other usual semantical notions such as model, validity, semantical consequence, etc., for $\mathcal{L}(M)$, are appropriately adapted from classical logic.

Grácio (1999) proved that structures of proper family of upper closed sets $\mathcal{A}^{\mathcal{F}}$ are sound and complete models for $\mathcal{L}(M)$.

2 About upper closed sets

In this section, we develop the notion of proper family of upper closed sets into a lattice and we call it upper closed set.

2.1 About lattices

We remember here some concepts about lattices.

A *lattice* is an algebraic structure determined by a non-empty set R and two binary operations $\wedge$ (meet) and $\vee$ (join) so that, for all $x, y, z \in R$, the following laws are valid:
R_1 $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ [commutativity];
R_2 $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ [associativity];
R_3 $(x \wedge y) \vee y = y$ and $(x \vee y) \wedge y = y$ [absorption].

We denote a lattice by $\mathbf{R} = (R, \wedge, \vee)$ and, for all lattices, the following laws are valid:
R_4 $x \wedge x = x$ and $x \vee x = x$ [idempotent];
R_5 $x \wedge y = x \Leftrightarrow x \vee y = y$ [ordering].

From law R_5, we have a very natural way to define a partial ordering relation in a lattice $\mathbf{R} = (R, \wedge, \vee)$, given by:

$$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$$

A lattice $\mathbf{R}$ can still be seen as an ordering structure $\mathbf{R} = (R, \leq)$ where $\leq$, the *ordering of the lattice*, is defined by means of one of the above equivalences.

In this case, for the ordered set $\mathbf{R} = (R, \leq)$ holds:

$$x \vee y = sup\{x, y\} \quad and \quad x \wedge y = inf\{x, y\}.$$

The following properties are also valid for meet and join:
R_6 $x \leq x \vee y$ and $y \leq x \vee y$;
R_7 $x \wedge y \leq x$ and $x \wedge y \leq y$;
R_8 $x \leq z$ and $y \leq z \Rightarrow x \vee y \leq z$;
R_9 $z \leq x$ and $z \leq y \Rightarrow z \leq x \wedge y$;
R_{10} $x \leq z$ and $y \leq w \Rightarrow x \vee y \leq z \vee w$;
R_{11} $x \leq z$ and $y \leq w \Rightarrow x \wedge y \leq z \wedge w$.

Let $\mathbf{R} = (R, \leq)$ be an ordered set such that for all $x, y \in R$ there are $inf\{x, y\}$ and $sup\{x, y\}$. The algebraic structure determined by $\mathbf{R} = (R, \wedge, \vee)$ in which

$$x \vee y = sup\{x, y\} \quad and \quad x \wedge y = inf\{x, y\}.$$

is a *lattice*.

A *distributive lattice* is a lattice $\mathbf{R} = (R, \wedge, \vee)$, in which the following distributive laws are valid for all $x, y, z \in R$:
R_{12} $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ [distributivity].

These distributive laws are known as distributive at right-hand side and, due to the commutative property, the distributive laws at left-hand side are also valid. Besides, only one of the laws listed above would be sufficient to characterize the commutative property.

If a lattice $\mathbf{R}$ has the least element given by the ordering $\leq$, then this element is the *zero* of $\mathbf{R}$ and is denoted by 0. On the other hand, if the lattice $\mathbf{R}$ has the greatest element, given by the ordering $\leq$, then this element is the *unit* of $\mathbf{R}$ and is denoted by 1.

If $\mathbf{R}$ has the zero, certainly $x \wedge 0 = 0$ and $x \vee 0 = x$ and, if it has the unit, then $x \wedge 1 = x$ and $x \vee 1 = 1$.

Let $\mathbf{R}$ be a lattice with 0 and 1. Given $x \in R$, an element $y \in R$ is the *complement* of x in $\mathbf{R}$ if $x \wedge y = 0$ and $x \vee y = 1$. A lattice $\mathbf{R}$ is *complete* when every element in R has its complement.

Let $\mathbf{R}$ be a lattice with 0 and 1. The complement of x, if it exists, is unique and it is denoted by $\sim x$.

A *Boolean algebra* $\mathbf{B}$ is a distributive and complete lattice.

A *homomorphism* from a lattice $\mathbf{R} = (R, \wedge, \vee)$ into a lattice $\mathbf{S} = (S, \wedge, \vee)$ is a function h from R into S such that:

$$h(x \wedge y) = h(x) \wedge h(y) \quad \textit{and} \quad h(x \vee y) = h(x) \vee (y).$$

Every homomorphism of lattices preserves ordering, that is:

$$x \leq y \Rightarrow h(x) \leq h(y).$$

A bijective homomorphism of lattices is called an *isomorphism*.

2.2 Upper closed set

Now, we envolve the concept of upper closed family into a lattice.

Let $(E, \leq)$ be a lattice. A non-empty set $S \in \mathcal{P}(E)$ is a *upper closed set* (ucs) if for all $a, b \in E$ the following condition holds:
(i) $a \in S$ and $a \leq b \Rightarrow b \in S$.

Proposition 2.1. *If the lattice* $(E, \leq)$ *has the element* 1*, then* 1 *belongs to every ucs in* $(E, \leq)$.
Proof: Every ucs is non-empty and for every $x \in E$*, we have that* $x \leq 1$. ■

Proposition 2.2. *The condition (i) above is equivalent to any of the following conditions:*
[1] $a \in S$ *and* $b \in E \Rightarrow a \vee b \in S$;
[2] $a \wedge b \in S \Rightarrow a \in S$ *and* $b \in S$.
Proof: (*i*) $\Leftrightarrow$ [1]: ($\Rightarrow$) *Let* $a \in S$ *and* $b \in E$. *Since* $a \leq a \vee b$, *by (i), we have that* $a \vee b \in S$. ($\Leftarrow$) *Let* $a \in S$ *and* $a \leq b$. *Thus,* $a \vee b = b$ *and by* $[1], b \in S$.
(*i*) $\Leftrightarrow$ [2]: ($\Rightarrow$) *If* $a \wedge b \in S$, *since* $a \wedge b \leq a$ *and* $a \wedge b \leq b$, *then by (i),* $a \in S$

and $b \in S$. *(*$\Leftarrow$*) If* $a \leq b$*, then* $a \wedge b = a$ *and since* $a \in S$*, then* $a \wedge b \in S$*. By* $[2], b \in S$. ■

Examples:
(a) If $(E, \leq)$ is a lattice, then E is an ucs in $(E, \leq)$. Given a set A, naturally $(\mathcal{P}(A), \subseteq)$ is a lattice in which meet coincides with the intersection of sets and join is the union of sets. In this lattice, $\mathcal{P}(A)$ is a upper closed set.
(b) Given $a \in (E, \leq)$, let $[a] = \{x \in E : a \leq x\}$. Then $[a]$ is a ucs generated by a and it is called *principal ucs.*
(c) If the lattice $(E, \leq)$ has the element 1, then the unitary set $\{1\}$ is a ucs, denoted by $\mathbf{1}$. The set $\mathbf{1}$ is a principal ucs generated by 1.

Proposition 2.3. *Let* $(E, \leq)$ *be a lattice. Then:*
(i) A finite intersection of ucs is also an ucs in $(E, \leq)$*;*
(ii) If $1 \in E$*, then any intersection of ucs is an ucs;*
(iii) Any union of ucs is an ucs.
Proof: (i) Let A *and* B *be two ucs in* $(E, \leq)$*. By definition,* A *and* B *are non-empty. Thus, if* $a \in A$ *and* $b \in B$*, certainly* $a \vee b \in A$*,* $a \vee b \in B$ *and* $A \cap B \neq \emptyset$*. Besides, for every* $c \in A \cap B$*, we have that* $c \in A$ *and* $c \in B$ *and, hence if* $c \leq d$*, then* $d \in A$ *and* $d \in B$*. Thus* $d \in A \cap B$ *and therefore,* $A \cap B$ *is an ucs.*
(ii) If $\{A_i\}_{i \in I}$ *is an ucs family, then* $1 \in A_i$*, for every* $i \in I$*. Therefore,* $1 \in \cap A_i$*, that is,* $\cap A_i \neq \emptyset$*. If* $a \in \cap A_i$ *and* $a \leq b$*, with* $b \in E$*, then for each* $i \in I$*,* $a \in A_i$*. Since* A_i *is an ucs and* $a \leq b$*, then* $b \in A_i$*. Thus,* $b \in \cap A_i$*.*
(iii) Let $\{A_i\}_{i \in I}$ *an ucs family. Consider* $a, b \in E$*, with* $a \in \cup A_i$ *and* $a \leq b$*. Then* $a \in A_i$*, for some* $i \in I$*, and since* A_i *is an ucs, then* $b \in A_i$*. Therefore,* $b \in \cup A_i$*.* ■

Example:
(a) The set of integers numbers $\mathbb{Z}$ determines a lattice in which $x \wedge y$ ($x \vee y$, respectively) is the least element (the greatest element, respectively) between x and y. Taking, for each $n \in \mathbb{Z}$, $A_n = \{m \in \mathbb{Z} : m \geq n\}$ we have that each A_n is an ucs and, also, $\cup A_n = \mathbb{Z}$ and $\cap A_n = \emptyset$.

Proposition 2.4. *Let* $(E, \leq)$ *and* $(F, \leq)$ *be two lattices and* $h : E \longrightarrow F$ *a homomorphism between lattices. Then:*
(i) If B *is an ucs of* $(F, \leq)$*, then* $h^{-1}(B)$ *is an ucs of* $(E, \leq)$*.*
(ii) If $(F, \leq)$ *has the element* 1*, then* $h^{-1}(1)$ *is an ucs of* $(E, \leq)$*.*
Proof: (i) Let $a \in h^{-1}(B)$ *and* $a \leq b$*. Since* h *is a homomorphism, then* $h(a) \leq h(b)$ *and* $h(a) \in B$*. Thus,* $h(b) \in B$ *and* $b \in h^{-1}(B)$*. Therefore,* $h^{-1}(B)$ *is an ucs. (ii) It is a particular case of (i) where* $B = \{1\}$*.* ■

Let $(E, \leq)$ be a lattice. For $\emptyset \neq A \subseteq E$, let: $[A] = \{x \in E : a \leq x,\ for\ some\ a \in A\}$.

Proposition 2.5. *The set $[A]$ is an ucs. It is the least ucs that contains A and it is the intersection of all ucs of $(E, \leq)$ that contain A.*
Proof: Let $x \in [A]$ and $x \leq y$. Thus, for some $a \in A$ we have that $a \leq x \leq y$. Therefore, $y \in [A]$ and $[A]$ is an ucs.
Besides, let B be an ucs such that $A \subseteq B$. If $x \in [A]$, then there is $a \in A$ such that $a \leq x$. Since $a \in B$, then $x \in B$. Thus, $[A] \subseteq B$ and $[A]$ is the intersection of all ucs that contain A. ■

An ucs A in $(E, \leq)$ is *proper* if $A \neq E$.

Examples:
(a) If A is a non-empty set, then set $\mathcal{P}(A) - \{\emptyset\}$ is a proper upper closed set in $(\mathcal{P}(A), \subseteq)$.
(b) If B is a collection of subsets of A such that $\emptyset \notin B$, then B can be extended to a proper upper closed set in $(\mathcal{P}(A), \subseteq)$, because $B \subseteq \mathcal{P}(A)$ is such that $\emptyset \notin B$. If $\mathbf{S} = [B] = \{C \in \mathcal{P}(A) :\ for\ some\ X \in B, X \subseteq C\}$, certainly $\mathbf{S}$ is a proper subset of $\mathcal{P}(A)$, because $\emptyset \notin \mathbf{S}$ and $\mathbf{S}$ is upper closed set and contains B.

Proposition 2.6. *If a lattice $(E, \leq)$ has the zero element, then A is a proper ucs if, and only if, A is an ucs and $0 \notin A$.*
Proof: ($\Leftarrow$) If $0 \notin A$ and A is a ucs, then $A \neq E$ and A is a proper ucs.
($\Rightarrow$) If $0 \in A$, since $0 \leq x$, for every $x \in E$, we have that $A = E$ and A is not proper. ■

An ucs A in a lattice $(E, \leq)$ is *maximal* when A is proper and it is not a proper subset of any other proper ucs.

In other words, a proper ucs is maximal if, and only if, it is a maximal element in the class of all proper ucs of $(E, \leq)$, with respect to inclusion $\subseteq$.
Examples:
(a) It is immediate that $E - \{0\}$ is the unique maximal ucs, when E has the zero element.
(b) The structure $(\mathbb{Z}, \leq)$ is a lattice that does not have maximal ucs, because if A is a proper ucs of $\mathbb{Z}$, then there is $n \in \mathbb{Z}$, such that $n \notin A$. Thus, A is a proper ucs of $A_n = \{m \in \mathbb{Z} : m \geq n\}$ that is a proper ucs of $\mathbb{Z}$.

An ucs A in a lattice $(E, \leq)$ is *irreducible* when A is proper and there are not ucs $B \neq A$ and $C \neq A$ such that $A = B \cap C$.

Then, every maximal ucs is irreducible.

Naturally, a ucs A is reducible when there are ucs B and C, distinct from A, such that $A = B \cap C$, and A is irreducible when it is not reducible.

Example:

(a) Let $E = \mathcal{P}(\{a, b, c\})$, $A = \{\{a, b\}, \{a, b, c\}\}$, $B = \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$, $C = \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$, $D = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The set D is irreducible, but it is not maximal; A is reducible because $A = B \cap C$.

Let F be a non-empty family of ucs over a lattice $(E, \leq)$. The inclusion relation of sets provides a partial ordering in $(F, \subseteq)$. A chain $\mathcal{C}$ of ucs is such that for every A, $B \in \mathcal{C}$ we have: $A \subseteq B$ or $B \subseteq A$.

Proposition 2.7. *If $0 \in E$, then any union of proper ucs is also a proper ucs.*

Proof: Since each member of the union does not have the 0, then their union does not have it either. ■

An ucs A in a lattice $(E, \leq)$ is *prime* when A is proper and for all $a, b \in E$ the following is valid:

$$a \vee b \in A \Rightarrow a \in A \text{ or } b \in A.$$

Proposition 2.8. *(i) Every maximal ucs in $(E, \leq)$ is irreducible;*

(ii) The ucs A is prime in $(E, \leq)$ if, and only if, A is irreducible.

Proof: (i) It is immediate.

(ii) $(\Rightarrow)$ According to the counter-positive, if A is a reducible ucs, then A is not a prime ucs. Let A reducible, that is, $A = B \cap C$ with B and C ucs and distinct from A, then $A \subset B$ and $A \subset C$. Then let $b \in B - A$ and $c \in C - A$. We have that $b \vee c \in B \cap C = A$, because B and C are ucs and, still, $b \leq b \vee c$ and $c \leq b \vee c$. Thus, A is not a prime ucs.

$(\Leftarrow)$ Also by the counter-positive, if A is not prime, then A is not irreducible. If A is not prime, exist $b, c \in E - A$ such that $b \vee c \in A$. Let $B = [A \cup \{b\}]$ and $C = [A \cup \{c\}]$. Clearly, $A \subseteq B \cap C$. If there is $x \in B \cap C - A$, then $b \leq x$ and $c \leq x$, therefore $b \vee c \leq x$. But $b \vee c \in A$ and A is an ucs. Consequently, $x \in A$, which is a contradiction. Thus, there is not $x \in B \cap C - A$ and then $A = B \cap C$, that is, A is reducible. ■

Proposition 2.9. *Let $(E, \leq)$ be a lattice with the zero element. Then, every proper ucs of $(E, \leq)$ is contained in a maximal ucs.*
Proof: It follows from the fact that $E - \{0\}$ is the only maximal ucs. ■

Proposition 2.10. *Let $(E, \leq)$ be a lattice with the zero element. For each element $a \neq 0$ in E, exists a maximal ucs A of $(E, \leq)$ such that $a \in A$.*
Proof: Clearly, $a \in E - \{0\}$, the only maximal ucs. ■

2.3 Lower closed set

In this section we deal with the dual concept of upper closed set.

Let $(E, \leq)$ be a lattice. A non-empty set $I \in \mathcal{P}(E)$ is a *lower closed set* (lcs) when for every $a, b \in E$ the following condition is valid:
(i) $b \in I$ and $a \leq b \Rightarrow a \in I$.

The results of the preceding section are applicable to the lower closed set concept and they will not be developed here. Naturally, this structure, with some adequacy, provide us a mathematical notion for 'few'.

2.4 Logical operator of consequence and ucs

We envolve here the ucs and the Tarski's operator.

A *consequence operator* over E is a function $C : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ such that, for every $A, B \subseteq E$:
(i) $A \subseteq C(A)$;
(ii) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$;
(iii) $C(C(A)) \subseteq C(A)$.

Examples:
(a) The examples that have motivated the definition of Tarski's consequence operator are the many monotonic logics, and distinct from classical logic, that emerged at the beginning of the twentieth century.
(b) The operator of topological closure, which for each set associates its closure, is another example of operator that admits the three conditions above.

From (i) and (ii), for each consequence operator C, follows that $C(C(A)) = C(A)$.

The consequence operator C over E is *finite* when, for every $A \subseteq E$, we have that $C(A) = \cup\{C(A_0) : A_0 \ is\ a\ finite\ subset\ of\ A\}$.

A *deductive system* is a pair (E, C) such that E is a set and C is a consequence operator over E.

Let C be a consequence operator over E. A subset A of E is *closed* in (E, C) when $C(A) = A$ and it is *open* when its complement, denoted by A^C, is closed.

The sets $C(\emptyset)$ and E are the smallest and the largest closed set of (E, C), respectively.

Let C and C_0 be two consequence operators over E. The operator C is *stronger* than C_0, what is denoted by $C_0 \preceq C$, if every closed set in relation to C is also closed in relation to C_0. In this case, C_0 is *weaker* than C.

Proposition 2.11. *Let C, C_0 be two consequence operators over E. The following clauses are equivalent:*

(i) C is stronger than C_0;
(ii) for every $A \subseteq X, C_0(C(A)) \subseteq C(A)$;
(iii) for every $A \subseteq X, C_0(A) \subseteq C(A)$.

Proof: *(i) $\Rightarrow$ (ii) If C is stronger than C_0, then for every closed set $A \subseteq E$ in C, $C(A)$ is closed in relation to C_0. Thus, $C_0(C(A)) = C(A)$. Then, $C_0(C(A)) \subseteq C(A)$.*
(ii) $\Rightarrow$ (iii) It is immediate, because $C_0(A) \subseteq C_0(C(A)) = C(A)$.
(iii) $\Rightarrow$ (i) If for every $A \subseteq E$, $C_0(A) \subseteq C(A)$, let $B \subseteq E$, B closed in relation to C, that is, $C(B) = B$. Then $C_0(B) \subseteq C(B) = B$. Since $B \subseteq C_0(B)$, then $B = C_0(B)$ and B is closed in relation to C_0. ■

The deductive system (E, C_1) is a *subsystem* of (F, C_2) if $E \subseteq F$ and $C_1 = C_2|_E$. We denote by $(E, C_1) \subseteq (F, C_2)$.

Proposition 2.12. *If $(E, C_1) \subseteq (F, C_2)$, then for every $A \subseteq E$, we have that $C_2(A) \subseteq E$.*
Proof: $C_1 = C_2|_E$. ■

A deductive system (E, C) is *vacuo* if $C(\emptyset) = \emptyset$.

Topological spaces are examples of vacuo deductive systems, but in this work, our real interest is about non-vacuo deductive systems.

A set $A \subseteq E$ is *dense* (*non-trivial*) when $C(A) = E$ and it is *non-dense* when $C(A) \neq E$. An element x is *dense* in (E, C) when $C(\{x\}) = E$.

Proposition 2.13. *(i) If A is dense and $A \subseteq B$, then B is dense.*
(ii) If A has some dense element, then it is dense.

Proposition 2.14. *Let $(E, \leq)$ be a lattice. The ucs family generated by subsets of $(E, \leq)$ naturally determines a consequence operator over E.*
Proof: Given $A \subseteq E$, consider $[A]$ the ucs generated by A. Certainly (i) $A \subseteq [A]$; (ii) $A \subseteq B \Rightarrow [A] \subseteq [B]$; and (iii) $[[A]] \subseteq [A]$. ■

By the Proposition 2.5, $[A] = \cap\{B : B\ is\ ucs\ of\ (E, \leq) \text{ and } A \subseteq B\}$.

2.5 Filters and ultrafilters

Here we particularize the notion of ucs.

Let $(E, \leq)$ be a lattice. A non-empty set $\mathcal{F} \subseteq \mathcal{P}(E)$ is a *filter* if for all $a, b \in E$ the following clause are valid:
(i) $a \in \mathcal{F}$ and $a \leq b \Rightarrow b \in \mathcal{F}$;
(ii) $a \in \mathcal{F}$ and $b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F}$.
A filter $\mathcal{F}$ is *proper* when $\mathcal{F} \neq \mathcal{P}(E)$. A filter $\mathcal{F}$ is *prime* when it is proper and for all $a, b \in E$ it is valid:
(iii) $a \vee b \in \mathcal{F} \Rightarrow a \in \mathcal{F}$ or $b \in \mathcal{F}$.

Examples:
(a) If A is a set with at least two distinct elements, for example $A = \{c, d\}$. The set $\mathcal{P}(A) - \{\emptyset\}$ is a proper ucs in $(\mathcal{P}(A), \subseteq)$. However, $\mathcal{P}(A) - \{\emptyset\}$ is not a filter because if we take $C = \{c\}$ and $D = \{d\}$, with $c \neq d$, hence $\emptyset \neq C \subseteq A$ and $\emptyset \neq D \subseteq A$, but $C \cap D = \emptyset \notin \mathcal{P}(A) - \{\emptyset\}$.
(b) Let U be an infinite set. The set $F = \{A \subseteq U : A\ is\ infinite\}$ is a upper closed set in $(U, \subseteq)$ but it is not a filter, because:
If $A \in F$ and $A \subseteq B \subseteq U$, then B is infinite and hence $B \in F$. Since U is infinite, there is an injective function $f : \mathbb{N} \longrightarrow U$. Let P be the set of even natural numbers and let I be the set of odd natural numbers. Thus, $f(P), f(I) \in F$, but $f(P) \cap f(I) = \emptyset \notin F$.

Proposition 2.15. *Let $(E, \leq)$ be a lattice. A non-empty set $\mathcal{F} \subseteq \mathcal{P}(E)$ is a filter if, and only if, $a \in \mathcal{F}$ and $b \in \mathcal{F} \Leftrightarrow a \wedge b \in \mathcal{F}$.*
Proof: ($\Rightarrow$) *If $\mathcal{F}$ is a filter, a, $b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F}$ and since $a \wedge b \leq a$, $a \wedge b \leq b$, $a \wedge b \in \mathcal{F} \Rightarrow a \in \mathcal{F}$ and $b \in \mathcal{F}$.*

($\Leftarrow$) If $a \in \mathcal{F}$ and $a \leq b$ then $a \wedge b = a \in \mathcal{F}$, therefore $b \in F$. If $a,\ b \in \mathcal{F}$ then $a \wedge b \in \mathcal{F}$. Hence, $\mathcal{F}$ is a filter. ■

Below we will treat with Boolean algebras. Since every Boolean algebra is a lattice, the theoretical developments performed so far remain valid.

Proposition 2.16. *Let $\mathbf{B} = (B, 0, 1, \sim, \wedge, \vee)$ be a Boolean algebra and $\mathcal{F}$ a filter in $\mathbf{B}$. Then the quotient algebra $\mathbf{B}|_{\mathcal{F}}$ in which: (i) $1_{\mathbf{B}|\mathcal{F}} = [1]$, (ii) $0_{\mathbf{B}|\mathcal{F}} = [0]$, (iii) $\sim [a] = [\sim a]$, (iv) $[a] \wedge [b] = [a \wedge b]$, (iv) $[a] \vee [b] = [a \vee b]$ is a Boolean algebra.*
Proof: See Rasiova and Sikorski (1968, p. 79). ■

An *ultrafilter* in a Boolean algebra $\mathbf{B}$ is a filter $\mathcal{U}$ such that, for every $a \in B$, either a or $\sim a$ belongs to $\mathcal{U}$.

Proposition 2.17. *Let $\mathbf{B} = (B, 0, 1, \sim, \wedge, \vee)$ a Boolean algebra. The following conditions are equivalent for every filter $\mathcal{F}$ in $\mathbf{B}$:*

(i) $\mathcal{F}$ is a ultrafilter;
(ii) $\mathcal{F}$ is a maximal filter;
(iii) $\mathcal{F}$ is a prime filter;
(iv) $\mathcal{F}$ is a irreducible filter;
(v) $\mathcal{F}$ is a proper filter and for all $a \in B$, either $a \in \mathcal{F}$ or $\sim a \in \mathcal{F}$;
(vi) $\mathbf{B}|_{\mathcal{F}}$ is the Boolean algebra $\mathbf{2} = \{0, 1\}$.

Proof: See Rasiowa (1974, p. 114). ■

Proposition 2.18. *Let $\mathbf{B} = (B, 0, 1, \sim, \wedge, \vee)$ be a Boolean algebra. Given $a, b \in B$, if $a \not\leq b$, then there is a ultrafilter $\mathcal{U} \in \mathbf{B}$ such that $a \in \mathcal{U}$ and $b \notin \mathcal{U}$.*
Proof: See Rasiowa and Sikorski (1968, p. 49). ■

Given a Boolean algebra $\mathbf{B} = (B, 0, 1, \sim, \wedge, \vee)$, let $\mathcal{U}(\mathbf{B})$ be the set of all ultrafilters in $\mathbf{B}$ and for each $a \in B$, let $h(a) = \{U \in \mathcal{U}(\mathbf{B}) : a \in U\}$ and $\mathbf{P}(\mathbf{B}) = \{h(a) : a \in B\}$.

Proposition 2.19. *If $\mathbf{B} = (B, 0, 1, \sim, \wedge, \vee)$ is a Boolean algebra, then h is a Boolean isomorphism from $\mathbf{B}$ onto $\mathbf{P}(\mathbf{B}) = \{h(a) : a \in B\}$.*
Proof: See Rasiowa and Sikorski (1968, p. 83-84). ■

In these conditions, $\mathcal{U}(\mathbf{B})$ is the Stone space of all ultrafilters of $\mathbf{B}$ and h is the Stone isomorphism from Boolean algebra $\mathbf{B}$ onto $\mathcal{U}(\mathbf{B})$.

3 A Propositional Logic of Many

In this section we introduce syntactic and semantics aspects of a propositional logic of many.

3.1 A propositional logic associated to many

To get the propositional logic of many, denoted by $\mathcal{L}(\star)$, we extend the classical propositional logic endowing the classical language $L(\neg, \wedge, \vee, \rightarrow)$ with a new operator $\star$. Formally, we denote the language of propositional logic of many by $L(\neg, \wedge, \vee, \rightarrow, \star)$. The propositional logic of many is determined by the following: the classical propositional tautologies, plus the following axioms for the operator $\star$:

(Ax_1) $\star(\varphi \vee \neg\varphi)$

(Ax_2) $\star\varphi \rightarrow \varphi$

(Ax_3) $\star(\varphi \wedge \psi) \rightarrow \star\varphi$;

the deduction rule *Modus Ponens* and by the rule:

$(R\star)$ $\vdash \varphi \leftrightarrow \psi \;/\; \vdash \star\varphi \leftrightarrow \star\psi$.

With the following intuition:

(Ax_1) Each theorem has many evidences;

(Ax_2) If φ has many evidences, then φ is not a contradiction;

(Ax_3) If $\varphi \wedge \psi$ have many evidences, then φ has many evidences.

$(R\star)$ If φ and ψ are equivalent, also are equivalent $\star\varphi$ and $\star\psi$.

These axioms and the rule $(R\star)$ try to include into the propositional context the fundamental conceptions of upper closed sets, introduced in the set theoretical environment.

Proposition 3.1. $\vdash \neg\star\bot$, *that is, a contradiction does not have many evidences.*

Proof: By (Ax_2) *we have* $\star\bot \rightarrow \bot$. *But as* $\vdash \neg\bot$, *then it follows that* $\vdash \neg\star\bot$. ■

Proposition 3.2. $\vdash \varphi \Rightarrow \vdash \star\varphi$.

Proof:

1. $\vdash \varphi$	*hypothesis*
2. $\vdash (\psi \vee \neg\psi) \rightarrow \varphi$	*CPC in 1*
3. $\vdash \varphi \rightarrow (\psi \vee \neg\psi)$	*CPC*
4. $\vdash (\psi \vee \neg\psi) \leftrightarrow \varphi$	*CPC in 2 and 3*
5. $\vdash \star(\psi \vee \neg\psi) \leftrightarrow \star\varphi$	*$R\star$ in 4*
6. $\vdash \star(\psi \vee \neg\psi) \rightarrow \star\varphi$	*CPC in 5*
7. $\vdash \star(\psi \vee \neg\psi)$	*Ax_1*
8. $\vdash \star\varphi$	*MP in 6 and 7.* ■

3.2 The algebra of many

We look for an algebraic model for $\mathcal{L}(\star)$.

An *algebra of many* is a 7-upla $\mathbf{M} = (M, 0, 1, \wedge, \vee, \sim, \sharp)$ such that the structure $(M, 0, 1, \wedge, \vee, \sim)$ is a Boolean algebra and $\sharp$ is the *many operator* which respects the following conditions:

(i) $\sharp 1 = 1$
(ii) $\sharp a \leq a$
(iii) $\sharp(a \wedge b) \leq \sharp a$.

From (iii), we have that $\sharp(a \wedge b) \leq \sharp a \wedge \sharp b$.

An element $0 \neq a \in M$ *has many evidences* when $\sharp a = a$.

From item (ii) it follows that $\sharp 0 = 0$. However, by definition 0 does not have many evidences. We did not include an algebraic axiom relative to $(R\star)$ because in the algebra always holds $a = b \Rightarrow \sharp a = \sharp b$.

An algebra $\mathbf{M}$ is *non-degenerate* when its universe M has at least two elements.

Theorem 3.3. *For each many algebra there exists a monomorphism h from $\mathbf{M}$ into a proper family of upper closed sets defined in $\mathcal{P}(\mathcal{P}(M))$.*
Proof: By the Stone's isomorphism, we know that for each Boolean algebra $(M, 0, 1, \vee, \wedge, \sim)$ there is a monomorphism h from it into a field of subsets of $\mathcal{P}(M)$. Denote the boolean algebra given by $Im(h)$ by $\mathbf{B} = (B, \emptyset, \cap, \cup, ^-)$, that is, h is an isomorphism from $\mathbf{M} = (M, 1, 0, \vee, \wedge, \sim, \sharp)$ to $\mathbf{B} = (B, \emptyset, \cap, \cup, ^-)$, where $B = h(M)$.

Next, we introduce a proper family of upper closed sets Ω in $\mathbf{B}$ and extend the isomorphism h to an isomorphism between $\mathbf{M}$ and $\mathbf{B} = (B, \emptyset, \cap, \cup, ^-, \Omega)$, in the following way. For each set $a \in M$ we define $h(a)^\sharp = h(\sharp a)$ and $\Omega = \{h(a) \in B : 0 \neq a = \sharp a\}$.

We need to show that Ω satisfies the conditions of the algebra of many:
(i) By definition $h(0) = \emptyset \notin \Omega$;
(ii) As $1 = \sharp 1$ and $h(1) = B$, then $B \in \Omega$;
(iii) If $a \in M$, since $\sharp a \leq a$, $h(a)^\sharp = h(\sharp a) \leq h(a)$;
(iv) If $a, b \in M$, since $\sharp(a \wedge b) \leq \sharp a \wedge \sharp b$, $(h(a) \cap h(b))^\sharp = h(a \wedge b)^\sharp = h(\sharp(a \wedge b)) \leq h(\sharp a \wedge \sharp b) = h(a)^\sharp \cap h(b)^\sharp$. ■

Proposition 3.4. *If $\mathbf{M} = (M, 1, 0, \vee, \wedge, \sim, \sharp)$ is an algebra of many and $a, b \in M$, then:*

(i) $b \leq a \Rightarrow \sharp b \leq \sharp a$;
(ii) $\sharp a \leq \sharp(a \vee b)$.

Proof: *(i)* $b \leq a \Rightarrow b = a \wedge b \Rightarrow \sharp b = \sharp(a \wedge b) \leq \sharp a$.
(ii) $a \leq a \vee b \Rightarrow \sharp a \leq \sharp(a \vee b)$. ■

3.3 The algebraic adequacy

We will indicate the set of propositional variables of $\mathfrak{L}(\star)$ by $Var\mathfrak{L}(\star)$, the set of its formulas by $For\mathfrak{L}(\star)$ and a generic algebra of many by $\mathcal{A}$.

The deduction of a formula ψ from Γ in $\mathfrak{L}(\star)$ is denoted by $\Gamma \vdash \psi$. When Γ is empty, the expression $\vdash \psi$ denotes that the formula ψ is a theorem of $\mathfrak{L}(\star)$.

A formula $\psi \in For\mathfrak{L}(\star)$ is *refutable* in Γ when $\Gamma \vdash \neg\psi$ holds, otherwise, ψ is *irrefutable.*

A *restrict valuation* is a function $\widetilde{v} : Var\mathfrak{L}(\star) \longrightarrow \mathcal{A}$, that interprets each variable of $\mathfrak{L}(\star)$ in an element of $\mathcal{A}$.

A *valuation* is a function $v : For\mathfrak{L}(\star) \longrightarrow \mathcal{A}$, that extends natural and uniquely $\widetilde{v}$ as follows:

$v(p) = \widetilde{v}(p)$
$v(\neg\varphi) = \sim v(\varphi)$
$v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$
$v(\varphi \rightarrow \psi) = v(\varphi) \rightarrowtail v(\psi)$
$v(\star\varphi) = \sharp v(\varphi)$.

As usual, operator symbols of the left sides represent logical operators and those in right sides represent algebraic operators.

Let $\mathcal{A}$ be an algebra of many. A valuation $v : For\mathfrak{L}(\star) \longrightarrow \mathcal{A}$ is a *model* for a set $\Gamma \subseteq For\mathfrak{L}(\star)$ when, for each formula $\varphi \in \Gamma$, $v(\varphi) = 1$.

In particular, a valuation $v : For\mathfrak{L}(\star) \longrightarrow \mathcal{A}$ is a model for $\varphi \in For\mathfrak{L}(\star)$ when $v(\varphi) = 1$.

A formula φ is *valid* in a algebra of many $\mathcal{A}$ when each valuation $v : For\mathfrak{L}(\star) \longrightarrow \mathcal{A}$ is a model for φ.

A formula φ is *many-valid*, what is denoted by $\vDash \varphi$, when it is valid in every algebra of many.

For $\Gamma \subseteq For(\mathcal{L}(\star))$, denoting the set of axioms of $\mathcal{L}(\star)$ by Ax, then $C(\Gamma) = \{\psi : \Gamma \cup Ax \vdash \psi\}$. We say that ψ is derivable in $\mathcal{L}(\star)$ or is a theorem of $\mathcal{L}(\star)$ when $\psi \in C(\emptyset)$, or, $\Gamma = \emptyset$.

A *theory* of $\mathcal{L}(\star)$ is a set $\Delta \subseteq For(\mathcal{L}(\star))$, such that $C(\Delta) = \Delta$.

When $\Delta = \emptyset$, we have the theorems of $\mathcal{L}(\star)$, that is, $\psi \in C(\emptyset) \Leftrightarrow \vdash \psi$.

Now let $(For\mathcal{L}(\star), \wedge, \vee, \rightarrow, \neg, \star, 0, 1)$ be the algebra of formulas of $\mathcal{L}(\star)$, such that $\wedge, \vee$ and $\rightarrow$ are binary operators, $\neg$ and $\star$ are unary operators, 0 and 1 are constants and $\varphi \rightarrow \psi =_{df} \neg\varphi \vee \psi$.

As usual, we define the Lindenbaum algebra of $\mathcal{L}(\star)$.
Given $\Gamma \subseteq For(\mathcal{L}(\star))$, we define an equivalence relation $\equiv$ by:

$$\varphi \equiv \psi \Leftrightarrow \ \Gamma \vdash \varphi \rightarrow \psi \ and \ \Gamma \vdash \psi \rightarrow \varphi.$$

The relation $\equiv$, more than an equivalence relation, is a congruence relation, since by rule $(R\star)$:

$$\varphi \equiv \psi \Leftrightarrow \ \Gamma \vdash \varphi \leftrightarrow \psi \Rightarrow \ \Gamma \vdash \star\varphi \leftrightarrow \star\psi \Leftrightarrow \star\varphi \equiv \star\psi.$$

For $\Gamma \cup \{\psi\} \subseteq For\mathcal{L}(\star)$, we denote by $[\psi]_\Gamma = \{\sigma \in For\mathcal{L}(\star) : \sigma \equiv \psi\}$ the equivalence class of ψ modulo $\equiv$ and Γ.

The *Lindenbaum algebra* of $\mathcal{L}(\star)$, denoted by $\mathcal{A}_\Gamma(\mathcal{L}(\star))$, is the quotient algebra defined by:
$\mathcal{A}_\Gamma(\mathcal{L}(\star)) = (For\mathcal{L}(\star)|_\equiv, 0, 1, \vee_\equiv, \neg_\equiv, \star_\equiv)$, such that:
$[\varphi] \vee_\equiv [\psi] = [\varphi \vee \psi]$,
$\neg_\equiv[\varphi] = [\neg\varphi]$,
$\star_\equiv[\varphi] = [\star\varphi]$,
$0 = [\varphi \wedge \neg\varphi] = [0]$ and
$1 = [\varphi \vee \neg\varphi] = [1]$.

In general, it will not be indicated the index $\equiv$ of operations.

When $\Gamma = \emptyset$ we denote the Lindenbaum algebra of $\mathcal{L}(\star)$ by $\mathcal{A}(\mathcal{L}(\star))$

Proposition 3.5. *In* $\mathcal{A}_\Gamma(\mathcal{L}(\star))$ *it is valid* $[\varphi] \leq [\psi] \Leftrightarrow \ \Gamma \vdash \varphi \rightarrow \psi$.
Proof: $[\varphi] \leq [\psi] \Leftrightarrow \ [\varphi] \vee [\psi] = [\psi] \Leftrightarrow [\varphi \vee \psi] = [\psi] \Leftrightarrow \ \Gamma \vdash \varphi \vee \psi \leftrightarrow \psi \Leftrightarrow \Gamma \vdash \varphi \rightarrow \psi$. ■

Proposition 3.6. *The algebra* $\mathcal{A}_\Gamma(\mathcal{L}(\star))$ *is an algebra of many.*
Proof: (Ax_1) $\star(\neg\varphi \vee \varphi) \Rightarrow [\star(\neg\varphi \vee \varphi)] = 1 \Rightarrow \sharp[\neg\varphi \vee \varphi] = 1$;
(Ax_2) $\star\varphi \to \varphi \Rightarrow [\star\varphi] \leq [\varphi] \Rightarrow \sharp[\varphi] \leq [\varphi]$;
(Ax_3) $\star(\varphi \wedge \psi) \to \star\varphi \Rightarrow [\star(\varphi \wedge \psi)] \leq [\star\varphi] \Rightarrow \sharp[\varphi \wedge \psi] \leq \sharp[\varphi]$;
$(R\star)$ $\{\vdash \varphi \leftrightarrow \psi \;/\vdash \star\varphi \leftrightarrow \star\psi\} : [\varphi \leftrightarrow \psi] = 1 \Rightarrow [\varphi] = [\psi] \Rightarrow \sharp[\varphi] = \sharp[\psi] \Rightarrow [\star\varphi] = [\star\psi] \Rightarrow [\star\varphi \leftrightarrow \star\psi] = 1.$ ■

The algebra $\mathcal{A}_\Gamma(\mathcal{L}(\star))$ is the *canonical model* of $\Gamma \subseteq \mathcal{L}(\star)$.

We will denote a valuation into the canonical model by $v_0 : For(\mathcal{L}(\star)) \longrightarrow \mathcal{A}_\Gamma(\mathcal{L}(\star))$. Of course, when $\Gamma = \emptyset$ we have $v_0 : For(\mathcal{L}(\star)) \longrightarrow \mathcal{A}(\mathcal{L}(\star))$.

Corollary 3.7. *Let* $\Gamma \cup \{\varphi\} \subseteq For(\mathcal{L}(\star))$. *If* $\Gamma \vdash \varphi$, *then* $[\varphi]$ *is the unit* 1 *in the model* $\mathcal{A}_\Gamma(\mathcal{L}(\star))$. *If the formula* φ *is irrefutable in* Γ, *then* $[\varphi] \neq 0$.
Proof: Let $\Gamma \vdash \varphi$. *Since* $\mathcal{A}_\Gamma(\mathcal{L}(\star))$ *always has an identity element* 1, *then:*

1. $\Gamma \vdash \varphi$ — *Hypothesis*
2. $\varphi \to (\psi \to \varphi)$ — *CPC*
3. $\varphi \to ((\varphi \to \varphi) \to \varphi)$ — *Substitution in 2*
4. $\Gamma \vdash (\varphi \to \varphi) \to \varphi$ — *MP in 1 and 3*

Hence: $1 = [\varphi \to \varphi] \leq [\varphi]$, *that is,* $[\varphi] = 1$.

On the other hand, when $[\varphi] = 1$, *then* $[\varphi \to \varphi] \leq [\varphi]$, *this means that holds* $\Gamma \vdash (\varphi \to \varphi) \to \varphi$. *Since* $\Gamma \vdash \varphi \to \varphi$, *by MP, it follows that* $\Gamma \vdash \varphi$. *Now,* φ *is irrefutable in* Γ *iff* $\Gamma \nvdash \neg\varphi$ *iff* $[\neg\varphi] \neq 1$ *iff* $\neg[\varphi] \neq 1$ *iff* $[\varphi] \neq 0$. ■

From preceding proposition and the definitions of 0 and 1 in the Lindenbaum algebra it results that for each formula φ:
$[\varphi] = 1$ iff $\Gamma \vdash \varphi$ and
$[\varphi] = 0$ iff $\Gamma \vdash \neg\varphi$.

Theorem 3.8. *(Soundness) The algebras of many are correct models for the logic* $\mathcal{L}(\star)$.
Proof: Let $\mathcal{A} = (A, 0, 1, \vee, \sim, \sharp)$ *be an algebra of many. It remains to prove that the axioms* $Ax_1 - Ax_3$ *are valid and the rule* $(R\star)$ *preserves validity:*
(Ax_1) $v(\star(\neg\varphi \vee \varphi)) = \sharp v(\neg\varphi \vee \varphi) = \sharp 1 = 1$;
(Ax_2) $v(\star\varphi \to \varphi) = v(\star\varphi) \rightarrowtail v(\varphi) = \sharp v(\varphi) \rightarrowtail v(\varphi) = \sim \sharp v(\varphi) \vee v(\varphi) = \sim \sharp v(\varphi) \vee (\sharp v(\varphi) \vee v(\varphi)) = (\sim \sharp v(\varphi) \vee \sharp v(\varphi)) \vee v(\varphi) = 1 \vee v(\varphi) = 1$;
(Ax_3) $v(\star(\varphi \wedge \psi) \to \star\varphi) = 1$, *because* $\sharp(v(\varphi) \wedge v(\psi)) \leq \sharp v(\varphi)$;
$(R\star)$ $v(\varphi \leftrightarrow \psi) = 1 \Rightarrow v(\varphi) = v(\psi) \Rightarrow \sharp v(\varphi) = \sharp v(\psi) \Rightarrow v(\star\varphi) = v(\star\psi) \Rightarrow v(\star\varphi \leftrightarrow \star\psi) = 1.$ ■

Corollary 3.9. *Propositional calculus $\mathcal{L}(\star)$ is consistent.*
Proof: Suppose that $\mathcal{L}(\star)$ is not consistent. Then there is $\varphi \in For\mathcal{L}(\star)$ such that $\vdash \varphi$ and $\vdash \neg\varphi$. By Soundness Theorem, φ and $\neg\varphi$ are valid formulas. Let v be a valuation in an algebra of many with two elements $2 = \{0, 1\}$. Since φ is valid, then $v(\varphi) = 1$ and $v(\neg\varphi) = \sim v(\varphi) = 0$. This contradict the fact of $\neg\varphi$ is valid. ■

Theorem 3.10. *Let $\varphi \in \mathcal{L}(\star)$. The following assertions are equivalent:*
(i) $\vdash \varphi$;
(ii) $\vDash \varphi$;
(iii) φ is valid in every algebra of many of sets $\mathbf{B} = (B, \emptyset, \cap, \cup, ^-, \Omega)$;
(iv) $v_0(\varphi) = 1$, where v_0 is the valuation of the canonical model $\mathcal{A}(\mathcal{L}(\star))$.
Proof: (i) $\Rightarrow$ (ii): it follows of Soundness Theorem.
(ii) $\Rightarrow$ (iii): is immediate.
(iii) $\Rightarrow$ (iv): since every algebra of many is isomorphic to an algebra of sets $\mathbf{B} = (B, \emptyset, \cap, \cup, ^-, \Omega)$ and $\mathcal{A}(\mathcal{L}(\star))$ is a algebra of many, the result follows.
(iv) $\Rightarrow$ (i): if $\varphi \in For\mathcal{L}(\star)$ and it is not derivable in $\mathcal{L}(\star)$, by Corollary 3.7, $[\varphi]$ do not coincide with the unity of $\mathcal{A}(\mathcal{L}(\star))$ and, thus $v_0(\varphi) \neq 1$. Therefore φ is not a valid formula. ■

Corollary 3.11. *(Completeness) For each $\varphi \in For\mathcal{L}(\star)$, if φ is valid, then φ is derivable in $\mathcal{L}(\star)$.*

3.4 Strongly completeness

This subsection shows the strongly adequacy of the algebraic models given by algebras of many.

As usual, $\Gamma \vDash \varphi$ denotes that every model of Γ is also a model of φ.

Proposition 3.12. *Let $\Gamma \subseteq For\mathcal{L}(\star)$. If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*
Proof: Let $v : Var\mathcal{L}(\star) \longrightarrow \mathcal{B}$ be a model for Γ. As in Theorem 3.8, rules of $\mathcal{L}(\star)$ preserve validity and if $v_{\mathcal{B}}(\psi) = 1$, for every $\psi \in \Gamma$, then $v_{\mathcal{B}}(\varphi) = 1$. ■

Proposition 3.13. *Let $\Gamma \subseteq For\mathcal{L}(\star)$ and $\mathcal{B}$ an algebra of many. If there is a model $v : For\mathcal{L}(\star) \longrightarrow \mathcal{B}$ for Γ, then Γ is consistent.*
Proof: Suppose that Γ is not consistent. Then $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ and so $v_{\mathcal{B}}(\varphi) = 1$ and $v_{\mathcal{B}}(\neg\varphi) = 1$. Since $v_{\mathcal{B}}(\neg\varphi) = 1$, it follows that $\sim v_{\mathcal{B}}(\varphi) = 1$ and, therefore, $v_{\mathcal{B}}(\varphi) = 0$ and we have a contradiction. ■

A model $v : For\mathcal{L}(\star) \longrightarrow \mathcal{B}$ is *adequate* for Γ when: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$.

Proposition 3.14. *If* $\Gamma \subseteq For\mathcal{L}(\star)$ *is consistent, then the canonical valuation is an adequate model to* Γ.
Proof: Considering the canonical valuation $v_0 : For\mathcal{L}(\star) \longrightarrow \mathcal{A}(\mathcal{L}(\star))$, $v_0(\varphi) = [\varphi]$, *by Corollary 3.7,* $v_0(\varphi) = 1$ *iff* $\Gamma \vdash \varphi$. *Therefore we have that* v_0 *is an adequate model to* Γ. ■

Theorem 3.15. *(Adequacy) Given* $\Gamma \subseteq For\mathcal{L}(\star)$, *the following conditions are equivalent:*

(i) Γ *is consistent;*

(ii) there is an adequate model to Γ*;*

(iii) there is an adequate model to Γ *in a algebra of many* $\mathcal{B}$ *that is an algebra of sets* $\mathbf{B} = (B, \emptyset, \cap, \cup, ^{-}, \Omega)$*;*

(iv) there is a model to Γ.

Proof: Proof: $(i) \Rightarrow (ii)$ *It follows of preceding proposition.*
$(ii) \Rightarrow (iii)$ *Since* $\mathcal{A}(\mathcal{L}(\star))$ *is an algebra of many and every algebra of many is isomorphic to an algebra of sets* $\mathbf{B} = (B, \emptyset, \cap, \cup, ^{-}, \Omega)$ *[Theorem 3.3], then the result follows.*
$(iii) \Rightarrow (iv)$ *It is a immediate consequence.*
$(vi) \Rightarrow (i)$ *It results directly by Proposition 3.13.* ■

Corollary 3.16. *Let* $\Gamma \cup \{\varphi\} \subseteq For\mathcal{L}(\star)$. *If* Γ *is consistent, the following conditions are equivalent:*

(i) $\Gamma \vdash \varphi$*;*

(ii) $\Gamma \vDash \varphi$*;*

(iii) every model of Γ *in a algebra of many of sets* $\mathbf{B} = (B, \emptyset, \cap, \cup, ^{-}, \Omega)$ *is a model to* φ*;*

(iv) $v_0(\varphi) = 1$ *for every canonical valuation* v_0 *into the canonical model* $\mathcal{A}(\mathcal{L}(\star))$.

4 Final considerations

We constructed a modal logic associated to the concept of many as introduced by Grácio (1999) and showed its adequacy relative to the class of algebras of many. However we believe that we can easily extend the Logic of Many to interpret aspects of filters and ultra-filters as well as compare the Logic of Many with other well known modal logics and determine some type of relational model to $\mathcal{L}(\star)$. Besides, it seems interesting to investigate aspects of 'few' in a formalized context.

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Bibliography

BELL, J. L.; MACHOVER, M. (1977) *A course in mathematical logic.* Amsterdam: North-Holland.

BARWISE, J.; COOPER, R. (1981) Generalized quantifiers and natural language. *Linguistics and Philosophy*, v. 4, 159-219.

BLACKBURN, P.; RIJKE, M.; VENEMA, Y. (2001) *Modal logic.* Cambridge: Cambridge University Press.

CARNIELLI, W. A.; PIZZI, C. (2001) *Modalità e multimodalità.* Milano: Franco Angeli.

CARNIELLI, W. A.; VELOSO, P. A. S. (1997) Ultrafilter logic and generic reasoning. In: *Proceedings of Kurt Gödel Colloquium*, 5, 1997, Berlin. Berlin: Springer-Verlag. p. 34-53.

CHAGROV, A.; ZAKHARYASCHEV, M. (1997) *Modal logic.* Oxford: Clarendon Press.

EBBINGHAUS, H. D. (1985) Extended logics: the generalized framework. In: BARWISE, J., FEFERMAN, S. (Ed.) *Model-theoretic logics.* Berlin: Springer-Verlag. p.25-76.

EBBINGHAUS, H. D.; FLUM, J.; THOMAS, W. (1984) *Mathematical logic.* New York: Springer-Verlag.

FEITOSA, H. A.; GRÁCIO, M. C. C.; NASCIMENTO, M. C. (2007) *A propositional logic for Tarski's consequence operator.* Campinas: CLE E-prints, p. 1-13.

GRÁCIO, M. C. C. (1999) *Lógicas moduladas e raciocínio sob incerteza.* Doctor Thesis (in Portuguese), Institute of Philosophy and Human Sciences, State University of Campinas, Campinas, Brazil. 194 p.

FITTING, M.; MENDELSOHN, R. L. (1998) *First-order modal logic.* Dordrecht: Kluwer.

HAMILTON, A. G. (1978) *Logic for mathematicians.* Cambridge: Cambridge University Press.

MENDELSON, E. (1987) *Introduction to mathematical logic.* 3. ed. Monterey, CA: Wadsworth & Brooks/Cole Advanced Books & Software.

MIRAGLIA, F. (1987) *Cálculo proposicional: uma interação da álgebra e da lógica.* Campinas: UNICAMP/CLE. (Coleção CLE, v. 1)

NASCIMENTO, M. C.; FEITOSA, H. A. (2005) As álgebras dos operadores de conseqüência. São Paulo: *Revista de Matemática e Estatística*, v. 23, n. 1, p. 19-30.

RASIOWA, H. (1974) *An algebraic approach to non-classical logics.* Amsterdam: North-Holland.

RASIOWA, H.; SIKORSKI, R. (1968) *The mathematics of metamathematics.* 2. ed. Waszawa: PWN - Polish Scientific Publishers.

SETTE, A. M., CARNIELLI, W. A., VELOSO, P. (1999) An alternative view of default reasoning and its logic. In: HAUESLER, E. H., PEREIRA, L. C. (Eds.) *Pratica*: Proofs, types and categories. Rio de Janeiro: PUC. p. 127-58.

VICKERS, S. (1990) *Topology via logic.* Cambridge: Cambridge University Press.

WÓJCICKI, R. (1988) *Theory of logical calculi:* basic theory of consequence operations. Dordrecht: Kluwer, 1988. (Synthese Library, v. 199)